Inventiones math. 59, 59–65 (1980)

Inventiones mathematicae

Universal Connections

Roger Schlafly

Department of Mathematics, University of California, Berkeley, CA 94720, USA

Let H be a compact Lie group. It is well-known [5] that principal H-bundles over a manifold M are classified by homotopy classes of maps into a certain space BH called the classifying space. In particular, every principal H-bundle over M is the pull-back of the canonical bundle over BH by some map from M into BH. In [4], Narasimhan and Ramanan showed that the canonical bundle over BH has a connection which is universal in the sense that any connection on any principal H-boundle over M may be induced by some map from M to BH.

A simple proof of the Narasimhan-Ramanan theorem will be given in this paper. The method yields the following generalizations: If the connection on the H-bundle is real-analytic, then the map into BH may be chosen to be real-analytic. If the connection is invariant under some compact Lie group G, then the map into BH may be chosen to be equivariant with respect to G. Furthermore, these maps are unique up to a connection preserving homotopy.

§1. The Orthogonal Group

The classifying space for $O(k)$ is the direct limit (as n tends to infinity) of the Grassmanian manifolds $G_k(\mathbb{R}^n) = O(n)/O(k) \times O(n-k)$. The total space of the canonical principal $O(k)$-bundle on $G_k(\mathbb{R}^n)$ is the Stiefel manifold $O(n)/O(n-k)$. The left-invariant (Maurer-Cartan) form $A^{-1}\,dA$ on $O(n)$ determines a k-by-k submatrix of 1-forms ω which is well-defined on $O(n)/O(n-k)$ and is called the canonical connection (see [2]). This connection is compatible with the direct limit.

This can be described more geometrically in terms of the associated vector bundle $\pi\colon \gamma^k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$. We have

$$G_k(\mathbb{R}^n) = \{W \subset \mathbb{R}^n \mid W \text{ is a subspace of dimension } k\},$$
$$\gamma^k(\mathbb{R}^n) = \{(v, W) \mid v \in W \quad \text{and} \quad W \in G_k(\mathbb{R}^n)\}$$

with $\pi(v, W) = W$. The horizontal curves through (v, W) are given by "rotations of

0020-9910/80/0059/0059/$01.40

v perpendicular to W," that is

$$t \mapsto (\exp(tB)\,v, \exp(tB)\,W)$$

where B belongs to the subspace of $o(n)$ that can be identified with Hom $(W, W^{\perp})$.

The proof of the Narasimhan-Ramanan theorem is motivated by the following well-known

Proposition. *Given a submanifold M^k of $\mathbb{R}^n$, the connection on M induced from $\mathbb{R}^n$ is the same as the one pulled back from $\gamma^k(\mathbb{R}^n)$ via the Gauss map $M \to G_k(\mathbb{R}^n)$.*

Proof. On a small neighborhood in $\mathbb{R}^n$ take an orthonormal (adapted) moving frame $v_1, \ldots, v_n$ such that, at points in M, $v_1, \ldots, v_k$ are tangent to M. There is a unique matrix of 1-forms $\omega = (\omega_{ij})$ with $dv_j = \sum_{i=1}^n v_i\,\omega_{ij}$. Let $A = (v_1, \ldots, v_n) \in O(n)$ so, in matrix notation, $dA = A \cdot \omega$ or $\omega = A^{-1}\,dA$. Thus $(\omega_{ij})_{i,j=1,\ldots,k}$ is just the pull-back of the universal connection with respect to the moving frame. But this agrees with the connection induced from $\mathbb{R}^n$, since on M we have $\nabla v_j = \sum_{i=1}^k v_i\,\omega_{ij}$.

Theorem. *Let M be a smooth m-dimensional manifold with a k-dimensional vector bundle E. Suppose that E has a (fiber) metric h and a connection compatible with h. Then for $n \geqq \frac{1}{2}(k+m)^2 + 7(k+m) + 10)$ there exists a smooth map $f\colon M \to G_k(\mathbb{R}^n)$ such that E and $f^*\gamma^k(\mathbb{R}^n)$ are isomorphic as bundles with metrics and connections. If the metric and connection on E are real-analytic then the map f may be taken to be analytic.*

Proof. Let g be any Riemannian metric on M. (If M is analytic, we may choose g to be analytic.) Make the total space of E into a Riemannian manifold by taking the orthogonal direct sum of g and h; i.e. use g on horizontal vectors and h on vertical vectors. By the Nash imbedding theorem E imbeds isometrically in $\mathbb{R}^n$ where, according to Gromov-Rokhlin [1], we may take $n = \frac{1}{2}((k+m)^2 + 7(k+m) + 10)$. (This imbedding may be taken to be analytic if E is analytic.) Define a map $E \to G_k(\mathbb{R}^n)$ as follows: Given a point in E, take the tangent space (translated to the origin) to the fiber at its image in $\mathbb{R}^n$. Composing this with the zero section gives the desired map.

We must now show that the connection pulled back from the universal connection is the same as the connection that we started with. By the argument of the above proposition, the pull-back connection is the same as the one induced from $\mathbb{R}^n$, and hence the same as the one induced from the Riemannian structure of E. The proof is completed by the following proposition which shows that, as vector bundles with metrics and connections, E is canonically identified with the normal bundle to the zero-section.

Proposition. *Let $\tilde{\nabla}$ be the Riemannian connection on E. Then for $X \in TM \subset TE$,*

(1) if Y is tangent to M, so is $\tilde{\nabla}_X Y$;

(2) if Y is normal to M, so is $\tilde{\nabla}_X Y$;

(3) the induced connections on the tangent and normal bundles to M agree with the original connections.

Proof. Let $\{x^i\}$ be a local coordinate system on M and let $\{e_\alpha\}$ be an orthonormal moving frame of local sections to the bundle $E\to M$. We get local coordinates on E by letting $(x^1(x),\ldots,x^m(x),y^1,\ldots,y^k)$ represent the point $\sum y^\alpha e_\alpha(x)\in E$. Let $\{\Gamma^\alpha_{i\beta}\}$ be the connection coefficients of the given connection ∇ on E, so $\nabla_{\partial/\partial x^i} e_\beta=\sum_{\alpha=1}^{k}\Gamma^\alpha_{i\beta}e_\alpha$. Note that $\Gamma_{\alpha i\beta}=\Gamma^\alpha_{i\beta}=-\Gamma^\beta_{i\alpha}$ since the frame $\{e_\alpha\}$ is orthonormal. A vector

$$\sum a^i\frac{\partial}{\partial x^i}+\sum b^\alpha\frac{\partial}{\partial y^\alpha}$$

in TE has horizontal component

$$\sum a^i\frac{\partial}{\partial x^i}+\sum a^i y^\beta\Gamma^\gamma_{i\beta}\frac{\partial}{\partial y^\gamma}$$

so the Riemannian metric on E is given by

$$\sum g_{ij}\,dx^i\,dx^j+\sum\delta_{\alpha\beta}\,dy^\alpha\,dy^\beta+\sum y^\beta\Gamma_{\alpha i\beta}\,dx^i\,dy^\alpha.$$

Let $\tilde\Gamma$ be the Levi-Civita connection for this metric, restricted to M. Then, by direct computation,

$$\tilde\Gamma_{ijk}=\tfrac12(\partial g_{ij}/\partial x^k+\partial g_{ik}/\partial x^j-\partial g_{jk}/\partial x^i),$$
$$\tilde\Gamma_{\alpha i\beta}=\Gamma_{\alpha i\beta},$$
$$\tilde\Gamma_{\alpha ij}=0=\Gamma_{ij\alpha}.$$

The result follows.

§2. The Narasimhan-Ramanan Theorem

Let H be any compact Lie group, which we may suppose to be imbedded in $O(k)$ for some k. Let $\pi\colon O(k)\to h$ be the projection determined by the Killing form. Let ω be the 1-form on $O(n)/O(n-k)$ with values in $o(k)$ which was defined in §1. Then $\pi\circ\omega$ is a connection on the principal H-bundle

$$O(n)/O(n-k)\to O(n)/H\times O(n-k).$$

Taking the direct limit as n tends to infinity gives the classifying space BH along with its canonical principal H-bundle and its canonical connection.

Theorem. *Suppose that* $\dim M\leqq m$ *and* $n\geqq\frac12((k+m)^2+7(k+m)+10)$. *Then every principal H-bundle with connection over M may be obtained as the pull-back of the canonical principal H-bundle and canonical connection over* $O(n)/H\times O(n-k)$ *by some map* $f\colon M\to O(n)/H\times O(n-k)$. *If the bundle with connection over M is real-analytic, then the map f may be taken to be analytic.*

Proof. When $H=O(k)$, this theorem is equivalent to the theorem in §1. To deduce the general case, suppose $P\to M$ is a principal H-bundle with connection. Let $\tilde P=P\times_H O(k)$ be the corresponding principal $O(k)$-bundle. The induced connection on $\tilde P$ is the pull-back of ω under some $O(k)$-map. Thus we have a sequence of connection preserving H-maps

$$P\to\tilde P\to O(n)/O(n-k).$$

The pull-back of ω has values in $h \subseteq O(k)$, so we could use $\pi \circ \omega$ instead of ω. Dividing out by H gives the desired map from M to $O(n)/H \times O(n-k)$. For more details, see [4].

In this theorem the maps from M to $O(n)/H \times O(n-k)$ are unique in the sense that there is a connection preserving homotopy in $O(2n)/H \times O(2n-k)$ between any two such maps. Furthermore we have a natural one-to-once correspondence between equivalence classes of principal H-bundles with connections and connection preserving homotopy classes of maps from M to BH. The proof of these assertions follows from the methods of §4 where a slightly different construction of BH is used.

§3. An Equivariant Version of the Narisimhan-Ramanan Theorem

The following equivariant version of the Nash imbedding theorem was recently proved.

Theorem [3]. *If M is a compact Riemannian manifold (possibly with boundary) and G is a compact Lie group acting on M by isometries, then there is an orthogonal representation ρ of G on some Euclidean space $\mathbb{R}^n$ and an isometric imbedding of M into $\mathbb{R}^n$ which is equivariant with respect to ρ.*

Let G and H be compact Lie groups, and consider H to be a subgroup of some $O(k)$. A G-manifold is a smooth manifold with a smooth action by G. A G-map is a smooth map of G-manifolds which commutes with the action of G.

Definition. A (G, H)-bundle on a G-manifold M consists of
(1) a principal H-bundle $P \to M$;
(2) a connection on $P \to M$;
(3) an action of G on P such that
  (a) G commutes with H,
  (b) the induced action of G on M agrees with the original one,
  (c) G preserves the connection.

Examples. (1) A G-manifold M^m with a G-invariant metric has a natural $(G, O(m))$-bundle given by the bundle of orthonormal frames.

(2) The canonical bundle over $O(n)/H \times O(n-k)$ is a $(O(n), H)$-bundle.

(3) Any representation ρ: $G \to O(n)$ determines a canonical (G, H)-bundle over $O(n)/H \times O(n-k)$.

Theorem. *For any (G, H)-bundle on a compact G-manifold M, there exists a representation ρ: $G \to O(n)$ and a G-map f: $M \to O(n)/H \times O(n-k)$ such that the (G, H)-bundle on M is isomorphic to the (G, H)-bundle induced by f and ρ. If the (G, H)-bundle on M is real-analytic then f may be chosen to be analytic.*

Proof. By imbedding the (G, H)-bundle in a $(G, O(k))$-bundle, it will suffice to treat the case where $H = O(k)$. Let $P \to M$ be the given principal $O(k)$-bundle, and give M a G-invariant Riemannian metric. Let $E = P \times_{O(k)} \mathbb{R}^k$ be the associated vector bundle and let E_1 be the unit disc bundle. Defining a Riemannian metric on E as before, it is easily checked that G acts on E_1 by isometries. Then for some representation ρ: $G \to O(n)$ there is an equivariant isometric imbedding of E_1 into $\mathbb{R}^n$. As before define f: $M \to G_k(\mathbb{R}^n)$ by looking at the tangent k-plane to the fiber in $\mathbb{R}^n$. The method of §1 shows that f has the required properties.

§4. A Uniqueness Theorem

Let G be a compact Lie group and let $\mathscr{H}$ be $\ell^2(\mathbb{N})$, the real Hilbert space of square summable sequences. Let P_k be the set of k-tuples of orthonormal vectors in $\mathscr{H}$, with the obvious action of $O(k)$ and $O(\mathscr{H})$. If H is a closed subgroup of $O(k)$, let BH be P_k/H. With either the strong or the weak topology, BH is the classifying space for H. With the strong topology, BH is a Banach manifold. For any continuous representation of G on $\mathscr{H}$, BH has a canonical (G, H)-bundle.

In particular, if $H = O(k)$ then

$$BO(k) = \{W \subset \mathscr{H} \mid W \text{ is a subspace of dimension } k\}$$

and the tangent bundle is given by

$$T_w BO(k) = L(W, W^\perp)$$

where $L(W, W^\perp)$ is the space of linear maps from W to $W^\perp$.

Let M be a G-manifold. Two (G, H)-bundles are isomorphic if there exists a bundle isomorphism that preserves the connection and commutes with G. A one-parameter family of G-maps f_t: $M \to BH$ is a connection preserving G-homotopy if for each t_1 and t_2 the (G, H)-bundles induced by f_{t_1} and f_{t_2} are isomorphic. The main theorem of this section is:

Theorem. *The inducing process gives a one-to-one correspondence between isomorphism classes of (G, H)-bundles on M and connection preserving G-homotopy classes of maps from M to BH.*

Proof. We first prove the theorem when $H = O(k)$. It is necessary to show that any $(G, O(k))$-bundle on any G-manifold M is induced by a map into $BO(k)$, and that there is a connection preserving G-homotopy between any two such maps. The first assertion follows from the technique of §3 and the theorem that any G-manifold admits an equivariant isometric imbedding into $\mathscr{H}$ (see [3]).

Suppose we have smooth maps $f_0, f_1: M \to BO(k)$ which are equivariant with respect to representations ρ_0, ρ_1: $G \to O(\mathscr{H})$ and an isomorphism of the induced $(G, O(k))$-bundles. We may think of the $(G, O(k))$-bundles as vector bundles with fiber metrics, connections, and G-actions. Having an isomorphism of the induced $(G, O(k))$-bundles means that for each $x \in M$ we have an isometry

$$h(x): f_0(x) \to f_1(x)$$

where h depends smoothly on x. Furthermore h is equivariant, i.e.

$$h(\sigma x)\,\rho_0(\sigma) = \rho_1(\sigma)\,h(x), \qquad x \in M, \qquad \sigma \in G,$$

and h maps horizontal curves to horizontal curves. A curve $c(s)$ in the vector bundle induced by f_0 lies over a curve $x(s)$ in M if $c(s) \in f_0(x(s))$ for all s. $c(s)$ is horizontal if

$$\frac{d}{ds} c(s) \perp f_0(x(s))$$

for all s. The condition on h then requires that

$$\frac{d}{ds} h(x(s))\, c(s) \perp f_1(x(s)).$$

Define a partial isometry $A(t)\colon \mathscr{H} \to \mathscr{H}$ as follows. Let $A(O)$ be the identity. For $\frac{1}{n+1} \leqq t \leqq \frac{1}{n}$, define $\theta_n(t) = \frac{\pi}{2} n(n+1)t - \frac{\pi}{2} n$ and let

$$A(t)(a_0, a_1, \ldots) = (a_0, \ldots, a_{n-2}, b_0, c_0, b_1, c_1, \ldots)$$

where

$$b_i = a_{n-1+i} \cos \theta_n(t)$$
$$c_i = a_{n-1+i} \sin \theta_n(t).$$

$A(t)$ is continuous in t and satisfies

$$A(\tfrac{1}{2})(a_0, a_1, \ldots) = (a_0, 0, a_1, 0, \ldots),$$
$$A(1)(a_0, a_1, \ldots) = (0, a_0, 0, a_1, \ldots).$$

Define a representation μ_t to be the direct sum of the representation $A(t) \circ \rho_0 \circ A(t)^{-1}$ on $\operatorname{Ran} A(t)$ and the representation on $\operatorname{Ran} A(t)^{\perp}$ induced by ρ_1 and the isomorphism $\operatorname{Ran} A(t)^{\perp} \cong \mathscr{H}$ given by

$$(-1)^n(0, \ldots, 0, -a_{n-1} \sin \theta_n(t), a_{n-1} \cos \theta_n(t), -a_n \sin \theta_n(t),$$
$$a_n \cos \theta_n(t), \ldots) \mapsto (a_{n-1}, a_n, \ldots)$$

for $\frac{1}{n+1} \leqq t \leqq \frac{1}{n}$. This satisfies

$$\mu_t \circ A(t) = A(t) \circ \rho_0$$

and

$$\mu_1 \circ \mathrm{A}(\tfrac{1}{2}) = A(\tfrac{1}{2}) \circ \rho_1.$$

Each $A(t)$ is a partial isometry, so $A(t) f_0$ is a connection preserving G-homotopy, being equivariant with respect to μ_t. Similarly, f_1 is connection preserving G-homotopic to $A(\frac{1}{2}) f_1$, which is equivariant with respect to μ_1. Let

$$g_t(x) = [\cos t (A(1) + \sin t\, A(\tfrac{1}{2})\, h(x)] f_0(x).$$

This is equivariant with respect to μ_1, and is a connection preserving G-homotopy from $A(1) f_0$ to $A(\frac{1}{2}) f_1$. This completes the proof when $H = O(k)$.

Now suppose that H is an arbitrary closed subgroup of $O(k)$. A given (G, H)-bundle on M may be enlarged to a $(G, O(k))$-bundle, so it is induced by some G-map of M into BH as before. Furthermore, it is not hard to show that any connection preserving G-map from a (G, H)-bundle to BH must factor through this enlargement. The theorem follows easily.

References

1. Gromov, M.L., Rokhlin, V.A.: Embeddings and immersions in riemannian geometry. Russian Math. Surveys **25**, 1–57 (1970)
2. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, Volume II. New York: Interscience 1969
3. Moore, J.D., Schlafly, R.: On equivariant isometric embeddings. Math. Zeit. in press (1980)
4. Narasimhan, M.S., Ramanan, S.: Existence of universal connections. Amer. J. Math. **83**, 563–572 (1961)
5. Steenrod, N.: The Topology of Fibre Bundles. Princeton University Press 1951

Received September 8, 1979