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On Equivariant Isometric Embeddings

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1. Introduction

The Nash embedding theorem [13] asserts that any Riemannian manifold possesses an isometric embedding into a Euclidean space of sufficiently large dimension. This article is devoted to a proof of an equivariant version of Nash's theorem.

Main Theorem. *If M is a compact Riemannian manifold and G is a compact Lie group which acts on M by isometries, there is an orthogonal representation ρ of G on some Euclidean space $\mathbb{E}^N$ and an isometric embedding from M into $\mathbb{E}^N$ which is equivariant with respect to ρ.*

The representation ρ can be regarded as a Lie group homomorphism from G into the orthogonal group $O(N)$ which acts on $\mathbb{E}^N$ by rotations and reflections; a smooth map $X: M \to \mathbb{E}^N$ is equivariant with respect to ρ if and only if $X(\sigma p) = \rho(\sigma) X(p)$, for all $\sigma \in G$, $p \in M$.

The main theorem is true in both the C^∞ and real analytic categories. We will work in the C^∞ category for the time being, and return to the real analytic case in §4. Moreover, the theorem holds for manifolds with boundary.

The main analytic tool used by Nash to prove his isometric embedding theorem is an implicit function theorem based upon the Newton iteration method. The implicit function theorem applies to the equivariant case with virtually no change. In order to apply the implicit function theorem we need to approximate a given G-invariant Riemannian metric on M by a metric induced by an equivariant embedding; we will do this by using the theory of the Laplace operator on compact Riemannian manifolds.

According to Gromov and Rokhlin [7], any n-dimensional compact Riemannian manifold can be isometrically embedded in $\mathbb{E}^N$, where $N = (1/2)\, n(n+1) + 3n + 5$. No such universal bound is possible in the equivariant case, and in fact, given any positive integer N, it is possible to construct a left invariant

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metric on the group S^3 of unit quaternions which is not induced by any equivariant embedding in $\mathbb{E}^n$ for $n \leqq N$, as described in § 6 of this article.

Moreover, the equivariant isometric embedding theorem does not hold without the assumption that M be compact. Indeed, Bieberbach [1] shows that the Poincaré disc with the hyperbolic metric of constant curvature -1 together with the circle group of rotations about the origin possesses no equivariant isometric embedding in any finite-dimensional Euclidean space.

It suffices to prove the equivariant isometric embedding theorem in the special case where M is an n-dimensional sphere S^n with a Riemannian metric invariant under a Lie subgroup G of $O(n+1)$. Indeed, by a theorem of Mostow and Palais [2, p. 315], any compact G-manifold possesses an equivariant embedding in a sphere S^n of sufficiently large dimension, even if the manifold has boundary, and by a partition of unity argument one easily extends a G-invariant metric on M to a G-invariant metric on S^n which makes this equivariant embedding isometric.

It will be convenient to formulate the equivariant isometric embedding problem in terms of certain Fréchet spaces. If G is a given compact Lie group acting on a compact manifold M and $\rho\colon G \to O(N)$ is a given representation, let

$C^\infty(M, \mathbb{E}^N) = \{C^\infty \text{ maps } X\colon M \to \mathbb{E}^N\}$,

$C^\infty_{G,\rho}(M, \mathbb{E}^N) = \{X \in C^\infty(M, \mathbb{E}^N) \mid X \text{ is equivariant with respect to } \rho\}$,

$\mathrm{Met}^\infty(M) = \{C^\infty \text{ symmetric rank two covariant tensors on } M\}$,

$\mathrm{Met}^\infty_G(M) = \{g \in \mathrm{Met}^\infty(M) \mid g \text{ is } G\text{-invariant}\}$.

These vector spaces become Fréchet spaces with the usual family of C^k norms. (Notation: $\|X\|_k$ and $\|g\|_k$ will denote the C^k norms of elements $X \in C^\infty(M, \mathbb{E}^N)$, $g \in \mathrm{Met}^\infty(M)$.) Finally let

$C^\infty_G(M, \mathbb{E}^N) = \cup \{C^\infty_{G,\rho}(M, \mathbb{E}^N) \mid \rho\colon G \to O(N) \text{ a representation}\}$.

We define a map $F\colon C^\infty_G(M, \mathbb{E}^N) \to \mathrm{Met}^\infty_G(M)$ by letting $F(X)$ be the metric induced on M by X. In terms of local coordinates $(u^1, \ldots, u^n)$ defined on an open subset U of M,

$$F(X) \mid U = \sum_{i,j=1}^{n} \frac{\partial}{\partial u^i}(X) \cdot \frac{\partial}{\partial u^j}(X)\, du^i\, du^j.$$

To prove the theorem, we need to show that given a positive-definite $g \in \mathrm{Met}^\infty_G(M)$, there is some representation $\rho\colon G \to O(N)$ and some embedding $X \in C^\infty_{G,\rho}(M, \mathbb{E}^N)$ such that $F(X) = g$.

We say that an element $g \in \mathrm{Met}^\infty_G(M)$ is *realizable* if there is a mapping (not necessarily an embedding) $X \in C^\infty_G(M, \mathbb{E}^N)$ for some N such that $F(X) = g$. The set of realizable metrics is closed under addition and multiplication by positive scalars.

There are two steps to the proof of the equivariant isometric embedding theorem. The first step consists of constructing a specific "perturbable" embedding $X_0 \in C^\infty_G(M, \mathbb{E}^{N_1})$ such that if $g_0 = F(X_0)$, then any $g_1 \in \mathrm{Met}^\infty_G(M)$ which is sufficiently close to g_0 can be realized as $F(X_1)$ for some embedding $X_1 \in C_G^\infty(M, \mathbb{E}^{N_1})$. This step relies on the Nash implicit function theorem and our treatment of it will be based upon the work of Schwartz [14] and Sergeraert [15], [16]. The second step consists of showing that any given positive-definite $g \in \mathrm{Met}_G^\infty(M)$ can be approximated arbitrarily closely by realizable metrics.

The two steps of the proof are put together in the following way: If g is a positive-definite element of $\mathrm{Met}_G^\infty(M)$, choose a constant $c>0$ so that $g-cg_0$ is positive-definite, where g_0 is the metric induced by the perturbable embedding X_0 constructed in step 1. (The metric cg_0 will be induced by the perturbable embedding $\sqrt{c}\,X_0$.) Use step 2 to approximate $g-cg_0$ by a realizable metric g_2; $g_2=F(X_2)$ for some $X_2 \in C_G^\infty(M, \mathbb{E}^{N_2})$. Then $g-g_2$ will be close to cg_0 and hence by step 1, there will be an element $X_1 \in C_G^\infty(M, \mathbb{E}^{N_1})$ such that $F(X_1)=g-g_2$. Hence $X=(X_1, X_2)$ will be an embedding in $C_G^\infty(M, \mathbb{E}^N)$ (where $N=N_1+N_2$) such that $F(X)=g$.

The above theorem extends the main result of an earlier article [10].

The two steps of the proof of the main theorem will be given in §§2 and 3. In §4 we will discuss the modifications necessary for proving the real analytic version of the main theorem. In §5 we will show that even if M is not compact it possesses an equivariant isometric embedding into Hilbert space, or into a finite-dimensional pseudo-Euclidean space if it has finitely many orbit types. Finally, in §6 we will discuss nonexistence theorems, including an extension of Bieberbach's example mentioned above.

2. Step 1. Nash's Implicit Function Theorem

Except for a few details, this is just like the corresponding step in the non-equivariant case.

For each choice of representation ρ, the map $F\colon C_{G,\rho}^\infty(M, \mathbb{E}^N) \to \mathrm{Met}_G^\infty(M)$ defined previously is Fréchet differentiable. Indeed, if X, ΔX are elements of $C_{G,\rho}^\infty(M, \mathbb{E}^N)$,

$$\begin{aligned} F(X+\Delta X)\,|\,U = {} & \sum_{i,j=1}^n \frac{\partial}{\partial u^i}(X)\frac{\partial}{\partial u^j}(X)\,du^i\,du^j \\ & + \sum_{i,j=1}^n \left(\frac{\partial}{\partial u^i}(X)\frac{\partial}{\partial u^j}(\Delta X)+\frac{\partial}{\partial u^j}(X)\frac{\partial}{\partial u^i}(\Delta X)\right) du^i\,du^j \\ & + \sum_{i,j=1}^n \frac{\partial}{\partial u^i}(\Delta X)\frac{\partial}{\partial u^j}(\Delta X)\,du^i\,du^j, \end{aligned}$$

from which we see that the Fréchet derivatives of F are given by the formulae

$$F'(X)(\Delta X)\,|\,U = \sum_{i,j=1}^n \left(\frac{\partial}{\partial u^i}(X)\frac{\partial}{\partial u^j}(\Delta X)+\frac{\partial}{\partial u^j}(X)\frac{\partial}{\partial u^i}(\Delta X)\right) du^i\,du^j,$$

$$F''(X)(\Delta_1 X, \Delta_2 X)\,|\,U = 2\sum_{i,j=1}^n \frac{\partial}{\partial u^i}(\Delta_1 X)\frac{\partial}{\partial u^j}(\Delta_2 X),$$

$$F^{(n)}(X)=0 \quad \text{for } n \geq 3.$$

Let g be a given positive-definite element of $\mathrm{Met}_G^\infty(M)$ and suppose that we can find an embedding $X \in C^\infty_{G,\rho}(M, \mathbb{E}^N)$ such that $F(X)$ is close to g. We might then try to construct an embedding $X+\Delta X$ such that $F(X+\Delta X)$ is closer to g by solving the linearized equation

$$F'(X)(\Delta X) = \Delta g, \quad \text{where } \Delta g = g - F(X).$$

If

$$\Delta g \,|\, U = \sum_{i,j=1}^{n} (\Delta g)_{ij}\, du^i\, du^j,$$

the linearized equation is

$$\frac{\partial}{\partial u^i}(X) \cdot \frac{\partial}{\partial u^j}(\Delta X) + \frac{\partial}{\partial u^j}(X) \cdot \frac{\partial}{\partial u^i}(\Delta X) = (\Delta g)_{ij}. \tag{1}$$

Following Nash, we impose the additional condition that ΔX be perpendicular to $X(M)$:

$$\frac{\partial}{\partial u^i}(X) \cdot \Delta X = 0. \tag{2}$$

Integrating by parts shows that Eqs. (1) and (2) are equivalent to the linear system

$$\begin{aligned} &\frac{\partial}{\partial u^i}(X) \cdot (\Delta X) = 0 \\ &\frac{\partial^2}{\partial u^i \partial u^j}(X) \cdot (\Delta X) = -\frac{1}{2}(\Delta g)_{ij} \end{aligned} \tag{3}$$

Recall that we are given Δg and we wish to solve for ΔX. It is natural to restrict attention to embeddings for which this linear system is nondegenerate.

Definition. An embedding $X: M \to \mathbb{E}^N$ is said to be *perturbable* if for every $p \in M$ there are local coordinates $(u^1, \ldots, u^n)$ defined on some neighborhood U of p such that the matrix of column vectors

$$A(p) = \left(\frac{\partial}{\partial u^i}(X)(p), \frac{\partial^2}{\partial u^i \partial u^j}(X)(p) \right)$$

has rank $n + \frac{1}{2}n(n+1)$. (Note that this condition does not depend upon the choice of local coordinates.)

Any G-manifold M possesses a perturbable equivariant embedding into a Euclidean space of sufficiently large dimension. Indeed, by the equivariant embedding theorem of Mostow and Palais mentioned in the introduction, there is an equivariant embedding

$$X = (x^1, \ldots, x^m): M \to \mathbb{E}^m$$

into a Euclidean space of some dimension m, from which we can construct a perturbable equivariant embedding

$$\tilde{X}=(x^1,\dots,x^m,(x^1)^2,\sqrt{2}\,x^1x^2,\dots,\sqrt{2}\,x^1x^m,(x^2)^2,\dots,(x^m)^2)$$

into a Euclidean space of dimension $m+\frac{1}{2}m(m+1)$.

We can rewrite the linear system (3) in the form

$$({}^tA)(\Delta X)=\begin{pmatrix}0\\-\frac{1}{2}\Delta g\end{pmatrix}.$$

If X is perturbable, ${}^tAA$ will be a nonsingular square matrix and

$$\Delta X=A({}^tAA)^{-1}\begin{pmatrix}0\\-\frac{1}{2}\Delta g\end{pmatrix} \tag{4}$$

will be a solution to (3), and in fact the argument given in Schwartz [14, p. 48] or Greene [4, pp. 32, 33] shows that it is the unique solution of smallest length. Thus even though A is defined in terms of local coordinates $(u^1,\dots,u^n)$ on M, the solution ΔX given by (4) does not depend on the choice of local coordinates. ΔX is a globally defined smooth mapping from M into $\mathbb{E}^N$, which as we will next check, is also equivariant.

If $p\in M$ and $\sigma\in G$ we can choose coordinate systems $(u^1,\dots,u^n)$ defined on a neighborhood U of p and $(\bar{u}^1,\dots,\bar{u}^n)$ defined on σU so that

$$\bar{u}^i(\sigma q)=u^i(q) \qquad \text{for } q\in U.$$

Since X is equivariant the matrices

$$A(p)=\left(\frac{\partial}{\partial u^i}(X),\frac{\partial^2}{\partial u^i\,\partial u^j}(X)\right)(p),\ \bar{A}(\sigma p)=\left(\frac{\partial}{\partial \bar{u}^i}(X),\frac{\partial^2}{\partial \bar{u}^i\,\partial \bar{u}^j}(X)\right)(\sigma p)$$

are related by the equation

$$\bar{A}(\sigma p)=\rho(\sigma)\,A(p),$$

and since $\rho(\sigma)$ is orthogonal,

$$\begin{aligned}\Delta X(\sigma p)&=\bar{A}(\sigma p)({}^t\bar{A}(\sigma p)\,\bar{A}(\sigma p))^{-1}\begin{pmatrix}0\\-\frac{1}{2}\Delta g(\sigma p)\end{pmatrix}\\&=\rho(\sigma)\,A(p)({}^tA(p)\,A(p))^{-1}\begin{pmatrix}0\\-\frac{1}{2}\Delta g(p)\end{pmatrix}=\rho(\sigma)\,\Delta X(p).\end{aligned}$$

Thus ΔX is indeed an element of $C^\infty_{G,\rho}(M,\mathbb{E}^N)$.

Hence we can define

$$L(X)\colon \mathrm{Met}^\infty_G(M)\to C^\infty_{G,\rho}(M,\mathbb{E}^N) \quad \text{by} \quad L(X)(\Delta g)=A({}^tAA)^{-1}\begin{pmatrix}0\\-\frac{1}{2}\Delta g\end{pmatrix},$$

so that

$$F'(X) \circ L(X) = \text{identity}.$$

Suppose that X_0 is a fixed perturbable embedding with $F(X_0) = g_0$. If g is a metric which is close to g_0, we might try to solve the nonlinear equation $F(X) = g$ by the Newton iteration

$$X_n = X_{n-1} + L(X_{n-1})(g - F(X_{n-1})), \qquad n \geqq 1.$$

If the X_n's were to converge in the C^∞ topology to an element $X_\infty \in C^\infty_{G,\rho}(M, \mathbb{E}^N)$, it would follow immediately that $F(X_\infty) = g$, but unfortunately this straightforward iteration does not converge since the estimate we obtain for $L(X)$,

$$\|L(X)(g - F(X))\|_{k-2} \leqq M_k \|X\|_k^3 \|g - F(X)\|_k, \tag{5}$$

M_k a constant, involves a loss of two derivatives.

To circumvent this difficulty, we make use of smoothing operators $S(t)$: $C^\infty_{G,\rho}(M, \mathbb{E}^N) \to C^\infty_{G,\rho}(M, \mathbb{E}^N)$ which satisfy the estimates

$$\begin{aligned} &\|S(t)\, X\|_{k+l} \leqq A_{k,l}\, t^k \|X\|_l, \\ &\|(I - S(t))\, X\|_k \leqq B_{k,l}\, t^{-l} \|X\|_{k+l}, \end{aligned} \tag{6}$$

where $A_{k,l}$, $B_{k,l}$ are suitable constants. In the nonequivariant case, smoothing operators $\tilde{S}(t)$: $C^\infty(M, \mathbb{E}^N) \to C^\infty(M, \mathbb{E}^N)$ satisfying (6) are constructed by Lang [9] or Schwartz [14, pp. 38, 39]. In the equivariant case, we set $S(t) = P \circ \tilde{S}(t)$, where P: $C^\infty(M, \mathbb{E}^N) \to C^\infty_{G,\rho}(M, \mathbb{E}^N)$ is the projection operator defined by

$$P(X)(p) = \int_G \rho(\sigma^{-1})\, X(\sigma p)\, d\sigma,$$

the integral being taken with respect to the Haar measure on G. (It is easily checked that P is bounded in each C^k norm.) Once we have defined these smoothing operators we can consider the modified Newton iteration:

$$X_n = X_{n-1} + S(t_{n-1}) \circ L(X_{n-1})(g - F(X_{n-1})), \qquad n \geqq 1,$$

where $\{t_1, t_2, \ldots, t_{n-1}, t_n, \ldots\}$ is a suitably chosen increasing sequence of real numbers.

The remarkable fact is that the t_n's can be chosen so that the modified Newton iteration converges. This is essentially the content of Nash's implicit function theorem. It is easiest to prove convergence when it is assumed (without loss of generality) that M is the n-sphere and G is a subgroup of $O(n+1)$. We can use the fact that F and L are invariant under rotations of the sphere (just as Schwartz [14, pp. 40, 41] uses the translation-invariant nature of F and L on the torus in his treatment of the nonequivariant case) to establish estimates which are somewhat stronger than (5): If $\|\Delta X\|_2 < 1$,

$$\begin{aligned} &\|L(X_0 + \Delta X)\, \Delta g\|_{k-2} \leqq M_k(1 + \|\Delta X\|_k)\, \|\Delta g\|_k, \\ &\|L(X_0 + \Delta X)\, F(X_0 + \Delta X)\|_{k-2} \leqq M_k(1 + \|\Delta X\|_k), \end{aligned}$$

where X_0 is the perturbable embedding fixed earlier. These inequalities yield the hypotheses for the elegant version of the implicit function theorem proven in the short note of Sergeraert [15], which can be applied without any change to our situation to establish convergence. It follows from Sergeraert's theorem that there exist k and ε such that if $\|g-g_0\|_k<\varepsilon$, then g can be realized by an equivariant isometric embedding, and Step 1 is finished.

3. Step 2. Approximation by Realizable Metrics

Thus the proof of the theorem reduces to showing that any positive-definite G-invariant metric can be approximated by realizable metrics. It will suffice to show that if g is any positive-definite element of $\mathrm{Met}_G^\infty(M)$, k a positive integer, and ε a small positive number, there is a realizable metric $F(X)$ such that $\|g-F(X)\|_k<\varepsilon$.

It follows from the nonequivariant version of the Nash embedding theorem that any positive-definite metric g on M can be expressed as a finite sum

$$g=\sum_{i=1}^{n} dy^i\otimes dy^i,$$

where each $y^i\colon M\to R$ is a smooth function. (Instead of relying on the entire proof of the Nash embedding theorem the following argument could rest on the weaker assertion that g can be C^k-approximated by a sum $\sum dy^i\otimes dy^i$, an assertion which is proven in Nash [13, p. 58] or Greene [4, p. 39] by a direct construction.)

According to the theory of elliptic operators on compact manifolds (as presented in [12, Chap. 3] for example), each y^i can be approximated in the C^k topology by a finite linear combination of eigenfunctions for the Laplace operator on M. It follows that there exists a finite sum

$$f_i=\sum_{j=1}^{n_i} f_{ij},$$

where each f_{ij} is an element of some eigenspace V_{ij} for the Laplace operator, such that

$$\|dy^i\otimes dy^i-df_i\otimes df_i\|_k<\varepsilon/n,$$

and hence

$$\|g-\sum_{i=1} df_i\otimes df_i\|_k<\varepsilon.$$

Since G acts on M via isometries it acts on each eigenspace V_{ij} via a linear representation (as described in [17, p. 257]). Let V_i be the minimal G-invariant subspace of $\sum_{j=1}^{n_i} V_{ij}$ which contains f_i, and define a smooth equivariant map

$$X_i\colon M\to V_i^* \quad \text{by} \quad (X_i(p))(f)=f(p) \qquad \text{for } f\in V_i.$$

Here we are using the fact that V_i is a linear space of real-valued functions defined on M.

We now construct a suitable G-invariant Euclidean metric on V_i^*. By differentiation under the integral sign it can be shown that

$$P: \mathrm{Met}^\infty(M) \to \mathrm{Met}_G^\infty(M)$$

defined by

$$P(g)(p) = \int_G (\sigma^* g)(p)\, d\sigma,$$

where $d\sigma$ is Haar measure, is norm decreasing in each C^k norm. We have

$$P(df_i \otimes df_i) = \int_G \sigma^*(df_i \otimes df_i)\, d\sigma,$$

where f_i is regarded as a real-valued function on M. But f_i also determines a linear functional $\bar{f}_i$: $V_i^* \to \mathbb{R}$ which satisfies the equation $\bar{f}_i \circ X_i = f_i$. We give V_i^* the Euclidean metric

$$\bar{g} = \int_G \sigma^*(d\bar{f}_i \otimes d\bar{f}_i)\, d\sigma.$$

(This metric is positive definite because as σ ranges throughout G, $\sigma(f_i)$ generates V_i.) With this choice of Euclidean metric on V_i^* it is easily verified that

$$F(X_i) = P(df_i \otimes df_i).$$

The X_i's fit together to give an equivariant mapping $X = (X_1, \ldots, X_n)$ into a Euclidean space of large dimension such that

$$\begin{aligned} \|g - F(X)\|_k &= \|g - \textstyle\sum F(X_i)\|_k \\ &= \|g - \textstyle\sum P(df_i \otimes df_i)\|_k \leqq \|g - \textstyle\sum df_i \otimes df_i\|_k < \varepsilon. \end{aligned}$$

This shows that g can be C^k-approximated by a realizable G-invariant metric and finishes the proof of the main theorem.

4. Real Analytic Embeddings

If the Riemannian manifold M is real-analytic, then the embedding in the equivariant isometric embedding theorem may be chosen to be real-analytic. The proof of this follows, as before, from the following two steps.

Step 1. Given an analytic perturbable $X_1 \in C_{G,\rho}^\infty(M, \mathbb{E}^N)$ *and an analytic* $g \in \mathrm{Met}_G^\infty(M)$ *sufficiently close to* $F(X_1)$ *in the* C^∞ *topology, then there exists an analytic* $X \in C_{G,\rho}^\infty(M, \mathbb{E}^N)$ *such that* $F(X) = g$.

Step 2. The analytic realizable metrics are dense in $\mathrm{Met}_G^\infty(M)$.

For the proof of step 1, we follow the proof of the analytic isometric embedding theorem in Greene-Jacobowitz [5]. We remark that their proof also holds for compact manifolds with boundary. See also the more general treatment by Gromov [6].

As before we use the Newton iteration

$$X_n = X_{n-1} + L(X_{n-1})(g - F(X_{n-1}))$$

only this time we don't have to use smoothing operators. Each X_n is evidently equivariant (i.e., $X_n \in C^{\infty}_{G,\rho}(M, \mathbb{E}^N)$) and analytic. By extending everything to a complex analytic extension of M and using Cauchy estimates, Greene and Jacobowitz are able to show that

$$X = \lim_{n \to \infty} X_n$$

exists, is real-analytic, and satisfies $F(X) = g$. Step 1 follows from this.

If M has no boundary, the proof of step 2 is exactly as before, because the eigenfunctions of Δ on M are real-analytic. If M has boundary we can analytically continue M to some slightly larger open Riemannian manifold N. Let

$$M_\varepsilon = \{x \in N \mid \operatorname{dist}(x, M) \leqq \varepsilon\}$$

for small enough $\varepsilon > 0$ for this to be a compact manifold with boundary. We can suppose that the G-action has been analytically continued to M_ε. Note that M_ε is real-analytic, and G acts on it by isometries. The eigenfunctions of Δ on M_ε with Dirichlet boundary conditions are real-analytic. Furthermore linear combinations of eigenfunctions (when restricted to M) are dense in $C^k(M)$ for all k. The proof of step 2 now proceeds exactly as before.

5. Embeddings into Other Spaces

The examples in §6 show that we cannot drop the condition that M be compact in our main theorem. However, we do have the following

Theorem. *Let G be a compact Lie group acting by isometries on some Riemannian manifold M, where M may have boundary and need not be compact. Then there exists a continuous orthogonal representation ρ on some real separable Hilbert space $\mathscr{H}$ and an isometric embedding from M into $\mathscr{H}$ which is equivariant with respect to ρ.*

Proof. Let $f: M \to \mathbb{E}^N$ be an isometric embedding. Let $\mathscr{H}$ be the direct sum of N copies of $L^2(G)$, i.e.,

$$\mathscr{H} = \{\psi = (\psi_1, \ldots, \psi_n) \mid \text{each } \psi_i \in L^2(G)\},$$

where the integration is with respect to normalized Haar measure. Define a continuous orthogonal representation ρ of G on $\mathscr{H}$ by

$$(\rho(\sigma)\psi)(\tilde{\sigma}) = \psi(\tilde{\sigma}\sigma) \qquad \sigma, \tilde{\sigma} \in G, \psi \in \mathscr{H}.$$

For $x \in M$ and $\sigma \in G$ define $h\colon M \to \mathscr{H}$ by

$$h(x)(\sigma) = f(\sigma x).$$

h is obviously smooth and injective.

We claim that h is an equivariant isometric embedding. h is equivariant because for $x \in M$ and $\sigma, \tilde{\sigma} \in G$,

$$\begin{aligned} h(\sigma x)(\tilde{\sigma}) &= f(\tilde{\sigma}\sigma x) \\ &= h(x)(\tilde{\sigma}\sigma) \\ &= \{\rho(\sigma)[h(x)]\}(\tilde{\sigma}) \end{aligned}$$

so

$$h(\sigma x) = \rho(\sigma)\, h(x).$$

If $v \in TM$, then

$$h_*(v)(\sigma) = (f \circ \sigma)_*(v) = f_*(\sigma_*(v))$$

so

$$|h_*(v)(\sigma)| = |f_*(\sigma_*(v))| = |\sigma_*(v)| = |v|$$

and

$$\begin{aligned} \langle h_*(v), h_*(v)\rangle_{\mathscr{H}} &= \int_G |h_*(v)(\sigma)|^2 d\sigma \\ &= \int_G |v|^2 d\sigma \\ &= |v|^2. \end{aligned}$$

Hence h is isometric. This completes the proof.

We now discuss embeddings into pseudo-Euclidean space, i.e., $\mathbb{R}^N$ with an indefinite metric. It is shown in the next section that the Poincaré n-disc does not have an equivariant isometric embedding into Euclidean space. However it does have an equivariant isometric embedding into the pseudo-Euclidean space $(\mathbb{R}^{n+1}, dx_1^2 + \ldots + dx_n^2 - dx_{n+1}^2)$ as the hyperboloid $x_1^2 + \ldots + x_n^2 - x_{n+1}^2 = -1$.

Let G be a compact Lie group acting smoothly on some (not necessarily compact) manifold M. Recall that two orbits are said to be equivalent if the corresponding isotropy groups are conjugate. An orbit type is an equivalence class of orbits. The theorem of Mostow and Palais cited earlier possesses a noncompact version due to Mostow [11]:

M has an equivariant embedding into some Euclidean space if and only if the number of orbit types is finite.

Now suppose that M has a symmetric bilinear form g (such as a Riemannian or pseudo-Riemannian metric) and that the action of G preserves g. Then using Mostow's theorem and a trick due to Gromov [6] we now prove:

Theorem. *M has an equivariant isometric embedding into some pseudo-Euclidean space if and only if the number of orbit types is finite. In particular, if M is compact then M has an equivariant isometric embedding into some pseudo-Euclidean space.*

Proof. In view of Mostow's theorem, it is sufficient to construct an equivariant isometric embedding from a given equivariant embedding $X \in C^\infty_{G,\rho}(M, \mathbb{E}^N)$.

According to §2, we can suppose that X has been modified to make it perturbable.

In the notation of §2, let

$$g' = F(X) = \sum \frac{\partial X}{\partial u^i} \cdot \frac{\partial X}{\partial u^j} du^i du^j$$

be the metric on M induced by X, where $(u^1, \dots, u^n)$ are local coordinates on M. Let

$$Y = L(X)(g - g') \in C^{\infty}_{G,\rho}(M, \mathbb{E}^N).$$

Then

$$\begin{aligned} g - g' &= F'(X) \cdot L(X)(g - g') \\ &= F'(X)(Y) \\ &= \sum \left(\frac{\partial X}{\partial u^i} \cdot \frac{\partial Y}{\partial u^j} + \frac{\partial X}{\partial u^j} \cdot \frac{\partial Y}{\partial u^i} \right) du^i du^j \end{aligned}$$

and hence

$$\begin{aligned} F(X+Y) &= \sum \frac{\partial}{\partial u^i}(X+Y) \cdot \frac{\partial}{\partial u^j}(X+Y)\, du^i du^j \\ &= F(X) + g - g' + F(Y) \\ &= g + F(Y). \end{aligned}$$

If $|\cdot|$ is the usual norm on $\mathbb{R}^N$ then the map

$$(X+Y, Y)\colon\ (M, g) \to (\mathbb{R}^N, |\cdot|^2) \oplus (\mathbb{R}^N, -|\cdot|^2)$$

is an equivariant isometric embedding.

6. Nonexistence Theorems

Let G be a compact Lie group acting on a Riemannian manifold M by isometries. In this section we will show that, under certain circumstances, M may not have any equivariant isometric embedding into $\mathbb{E}^N$. This means that M has no isometric embedding into $\mathbb{E}^N$ which is equivariant with respect to some homomorphism from G to the isometry group of $\mathbb{E}^N$. However, any homomorphism $G \to \operatorname{Isom}(\mathbb{E}^N)$ is conjugate to a homomorphism $G \to O(N) \subset \operatorname{Isom}(\mathbb{E}^N)$, so it is sufficient to consider isometric embeddings which are equivariant with respect to some orthogonal representation of G.

Lemma. *Let $\{X_1, X_2, X_3\}$ be a basis for $\mathfrak{su}(2)$ satisfying*

$$[X_1, X_2] = X_3 \qquad [X_2, X_3] = X_1 \qquad [X_3, X_1] = X_2.$$

Then for any Lie algebra homomorphism $\rho\colon \mathfrak{su}(2) \to \mathfrak{u}(N)$ and any $v \in \mathbb{C}^N$ we have

$$|\rho(X_3)v|^2 \leqq \frac{N-1}{2}(|\rho(X_1)v|^2 + |\rho(X_2)v|^2).$$

Proof. It is sufficient to prove this for irreducible representations. So suppose that ρ is an irreducible representation of dimension $2l+1 \leqq N$. Then $\rho(X_3)$ has eigenvalues $-l\sqrt{-1}$, $(-l+1)\sqrt{-1}, \ldots, l\sqrt{-1}$, and the Casimir operator $\sum_{i=1}^{3} \rho(X_i)^2$ has the value $-l(l+1)$.

It follows that

$$|\rho(X_3)v|^2 \leqq l^2 |v|^2$$

and

$$\sum_{i=1}^{3} |\rho(X_i)v|^2 = l(l+1)|v|^2.$$

Hence

$$|\rho(X_1)v|^2 + |\rho(X_2)v|^2 \geqq l|v|^2 \geqq \frac{1}{l} |\rho(X_3)v|^2.$$

Theorem. *Let G be a non-abelian compact Lie group (with Lie algebra $\mathfrak{g}$) acting smoothly on M. Suppose that for some point $y \in M$, the derived map $\mathfrak{g} \to T_y M$ is injective. Then for any integer N there exists a G-invariant metric on M such that M has no equivariant isometric embedding into $\mathbb{E}^N$.*

Proof. By hypothesis, there exist nonzero X_1, X_2, $X_3 \in \mathfrak{g}$ satisfying

$$[X_1, X_2] = X_3 \qquad [X_2, X_3] = X_1 \qquad [X_3, X_1] = X_2.$$

Furthermore the vectors

$$\tilde{X}_i = \frac{d}{dt} e^{tX_i} \cdot y|_{t=0} \in T_y M$$

are linearly independent. Given N, put a metric on $\{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3\}$ that satisfies

$$|\tilde{X}_3|^2 > \frac{N-1}{2} (|\tilde{X}_1|^2 + |\tilde{X}_2|^2).$$

Extend this to a G-invariant metric on M.

Now suppose we had an isometric embedding $f\colon M \to \mathbb{E}^N$ which is equivariant with respect to $\rho\colon G \to O(N)$. Denote also by ρ the induced map $\mathfrak{g} \to \mathfrak{o}(N)$. Then

$$f_*(\tilde{X}_i) = \rho(X_i) f(y)$$

and hence

$$\begin{aligned} |\rho(X_3) f(y)|^2 &= |f_*(\tilde{X}_3)|^2 \\ &= |\tilde{X}_3|^2 \\ &> \frac{N-1}{2} (|\tilde{X}_1|^2 + |\tilde{X}_2|^2) \\ &= \frac{N-1}{2} (|f_*(\tilde{X}_1)|^2 + |f_*(\tilde{X}_2)|^2) \\ &= \frac{N-1}{2} (|\rho(X_1) f(y)|^2 + |\rho(X_2) f(y)|^2) \end{aligned}$$

which contradicts the lemma.

Thus left-invariant metrics on $SU(2)$ (or any other compact non-abelian Lie group) require arbitrarily many dimensions for an equivariant isometric embedding into Euclidean space.

In general, if the manifold M is not compact there may not be any equivariant isometric embedding of M into Euclidean space. There are examples of manifolds with compact Lie group actions having infinitely many orbit types. These manifolds do not even have equivariant embeddings into Euclidean space. A somewhat different example was provided by Bieberbach [1] who showed that the Poincaré disc with its obvious circle action has no equivariant isometric embedding into Euclidean space. The following is a mild generalization:

Theorem. *Let M be a complete simply-connected manifold with sectional curvatures $\leqq -\varepsilon < 0$, and let G be a compact connected Lie group acting nontrivially on M by isometries. Then M has no equivariant embedding into any Euclidean space.*

Proof. By a theorem of Cartan (see [8], p. 111) there exists a point $p \in M$ which is fixed by G. The Cartan-Hadamard theorem tells us that $\exp: T_pM \to M$ is a diffeomorphism. Without loss of generality, we may take the group G to be S^1.

Let $\gamma: [0, \infty) \to M$ be a geodesic ray (parametrized by arc-length) with $\gamma(0) = p$ which is not fixed by the S^1 action. Define a vector field V along γ by

$$V(t) = \frac{\partial}{\partial \alpha} e^{i\alpha} \gamma(t)|_{\alpha=0}, \qquad t \in [0, \infty), e^{i\alpha} \in S^1.$$

V is a Jacobi field along γ, and satisfies

$$V(0) = 0,$$
$$V'(0) \perp \gamma'(0),$$
$$|V'(0)| \neq 0.$$

We now consider the analogous set-up in M_0, the space form of constant curvature $-\varepsilon$. In geodesic coordinates about a point $p_0 \in M_0$, the metric on M_0 is

$$ds^2 = dr^2 + \frac{\sinh^2(\sqrt{\varepsilon}\, r)}{\sqrt{\varepsilon}} d\theta^2.$$

Here r is the distance to p_0 and θ is the angle about p_0. Let $\gamma_0: [0, \infty) \to M_0$ be a geodesic parametrized by arc-length with $\gamma_0(0) = p_0$. Let V_0 be the vector field $|V'(0)|\, \partial/\partial\theta$ restricted to γ_0. V_0 is a Jacobi field satisfying

$$V_0(0) = 0,$$
$$V_0'(0) \perp \gamma_0'(0),$$
$$|V_0'(0)| = |V'(0)|.$$

The above Jacobi fields satisfy the hypotheses of the Rauch comparison theorem (see [3]), so we may conclude that $|V(t)| \geqq |V_0(t)|$ for all $t \in [0, \infty)$. We easily compute

$$|V_0(t)| = |V'(0)| \frac{\sinh(\sqrt{\varepsilon}\, t)}{\sqrt{\varepsilon}}.$$

For $t \in [0, \infty)$, let C_t: $[0, 2\pi] \to M$ be the closed curve given by

$$C_t(\alpha) = e^{i\alpha} \cdot \gamma(t) \qquad 0 \leqq \alpha \leqq 2\pi.$$

The length of C_t is $2\pi |V'(t)|$, so

$$\text{length } (C_t) \geqq 2\pi |V'(0)| \frac{\sinh(\sqrt{\varepsilon}\, t)}{\sqrt{\varepsilon}}.$$

For the sake of obtaining a contradiction, suppose that f: $M \to \mathbb{E}^N$ is an isometric embedding which is equivariant with respect to

$$e^{i\alpha} \to e^{\alpha B} \qquad e^{i\alpha} \in S^1, B \in \mathfrak{o}(N).$$

Then the curve $f \circ C_t$ satisfies

$$(f \circ C_t)(\alpha) = e^{\alpha B} \cdot f(\gamma(t))$$

and has length

$$\begin{aligned} \text{length } (f \circ C_t) &= 2\pi |B f(\gamma(t))| \\ &\leqq 2\pi \|B\| (|f(p)| + t). \end{aligned}$$

But f is isometric, so combining this with the above inequality gives

$$2\pi |V'(0)| \cdot \frac{\sinh(\sqrt{\varepsilon}\, t)}{\sqrt{\varepsilon}} \leqq 2\pi \|B\| (|f(p)| + t)$$

which is a contradiction for large t.

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