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A Chern number for gauge fields on $\mathbb{R}^4$

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Let F be the curvature of some connection on some principal bundle over $\mathbb{R}^4$. I show that if F decays as fast as $(r^2\ln r)^{-1}$ as r tends to infinity, then

$$\frac{1}{8\pi^2}\int \mathrm{tr}F^2$$

is an integer. If F decays like r^{-2}, any real value is possible. There is an analogous statement for $\mathbb{R}^{2n}(n>2)$, although it fails for $\mathbb{R}^2$.

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1. INTRODUCTION

Let G be a compact Lie group and let ρ: $G\rightarrow \mathrm{U}(N)$ be a representation. Let F be the curvature of some connection on some principal G bundle over S^4. The Chern number of the associated vector bundle is

$$c_2=\frac{1}{8\pi^2}\int_{S^4}\mathrm{tr}\rho(F)^2.$$

According to the theory of characteristic classes, this quantity is always an integer.

Now consider the curvature F of a connection on a principal G bundle over $\mathbb{R}^4$. In this paper, we prove

Theorem 1.1: If

$$|F|\leqslant C/r^2\ln r,\quad r\geqslant 2$$

for some constant C, then

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\mathrm{tr}\rho(F)^2$$

is an integer.

This formula is of interest in quantum field theory because the curvature F is the same as a Yang–Mills (gauge) field over Euclidean space. For discussions of related matters, see Refs. 1, 2, and 3.

If G is abelian or $N=1$, the integer obtained in Theorem 1.1 must be zero. If $G=\mathrm{SU}(N)\,(N>1)$, any integer value is possible.

It is not known whether the hypothesis of Theorem 1.1 can be relaxed to the energy $\int_{\mathbb{R}^4}|F|^2$ being finite, although this is suggested by Refs. 2, 4, and 5. However, it is shown in Sec. 6 that $(1/8\pi^2)\int_{\mathbb{R}^4}\rho(F)^2$ may not be an integer if we assume only that $|F|\leqslant C/r^2$. This example is equivalent to the one given in Ref. 5, and has infinite energy.

Theorem 1.1 is proved in Secs. 2, 3, 4, and 5. Section 3 gives a holonomy formula similar to the one in Ref. 6 and may be read independently of the rest of the paper. Section 7 shows how to handle dimensions other than four.

2. OUTLINE OF THE PROOF

For the proof of Theorem 1.1 it suffices to consider the associated $\mathrm{U}(N)$ bundle. Hence we may, without loss of generality, take $G=\mathrm{U}(N)$ and suppress mention of ρ.

Relative to some trivilization of the bundle, the connection A is a one-form on $\mathbb{R}^4$ with values in $\mathrm{u}(N)$. The curvature

$$F=dA+A^2$$

is a two-form on $\mathbb{R}^4$ with values in $\mathrm{u}(N)$. The Chern–Weil formalism (Ref. 7, p. 114) tells us that

$$\mathrm{tr}F^2=d\,\mathrm{tr}(AF-\tfrac{1}{3}A^3).$$

Let S^3_r be the sphere of radius r in $\mathbb{R}^4$. If $|F|\leqslant C/r^2\ln r$. Then $\mathrm{tr}F^2$ is integrable on $\mathbb{R}^4$. By Stokes' theorem,

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\mathrm{tr}F^2=\lim_{r\to\infty}\frac{1}{8\pi^2}\int_{S^3_r}\mathrm{tr}(AF-\tfrac{1}{3}A^3).$$

Given $\epsilon>0$ and r sufficiently large, we will show that there exists a smooth map $\widetilde{T}$: $S^3_r\rightarrow\mathrm{U}(N)$ such that

$$\left|\frac{1}{8\pi^2}\int_{S^3_r}\mathrm{tr}[AF-\tfrac{1}{3}A^3-\tfrac{1}{3}(\widetilde{T}^{-1}d\widetilde{T})^3]\right|<\frac{\epsilon}{2}.$$

If we choose r large enough such that

$$\left|\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\mathrm{tr}F^2-\frac{1}{8\pi^2}\int_{S^3_r}\mathrm{tr}(AF-\tfrac{1}{3}A^3)\right|<\frac{\epsilon}{2},$$

then it will follow that

$$\left|\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\mathrm{tr}F^2-\frac{1}{24\pi^2}\int_{S^3_r}\mathrm{tr}(\widetilde{T}^{-1}d\widetilde{T})^3\right|<\epsilon.$$

We now recall a special case of Bott's work on periodicity in K-theory[8].

Theorem 2.1: For $N\geqslant 2$ we have an isomorphism $\pi_3\mathrm{U}(N)\simeq\mathbb{Z}$ given by assigning to a smooth map $\widetilde{T}$: $S^3\rightarrow\mathrm{U}(N)$ the integer

$$\frac{1}{24\pi^2}\int_{S^3}\mathrm{tr}(\widetilde{T}^{-1}d\widetilde{T})^3.$$

Thus for $\epsilon>0$ there exists an integer n such that

$$\left|\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\mathrm{tr}F^2-n\right|<\epsilon.$$

Hence $(1/8\pi^2)\int_{\mathbb{R}^4}\mathrm{tr}F^2$ must be an integer.

3. THE HOLONOMY PROPAGATOR

Let M be the sector in $\mathbb{R}^2$ described by polar coordinates (r,θ) with $0\leqslant r\leqslant r_0$ and $0\leqslant\theta\leqslant\theta_0$. Suppose we have a connection A on the trivial principal G bundle. For this paper we can assume that G is $\mathrm{U}(N)$, although Theorems 3.1 and 3.2 hold for any Lie group. A is a one-form on M with values in

$\mathfrak{u}(N)$. The curvature is $F = dA + A^2$.

For $x \in M$ let $c_x : [0, 1] \to M$ be the curve

$$c_x(t) = tx, \quad 0 \leqslant t \leqslant 1.$$

A section $u: M \to \mathbb{C}^N$ of the associated vector bundle is parallel along c_x iff it satisfies

$$c_x^*(du + Au) = 0.$$

The fundamental solution to this differential equation is the path ordered exponential

$$P\exp\left(-\int_{c_x} A\right).$$

If u is parallel along c_x then

$$u(x) = P\exp\left(-\int_{c_x} A \cdot u(0)\right).$$

$P\exp - (\int_{c_x} A)$ is called the holonomy from 0 to x along c_x. It may also be defined as a product integral (Ref. 9, p. 15).

Theorem 3.1: If

$$T(x) = P\exp\left(-\int_{c_x} A\right)$$

then

$$A(x) = -dT\,T^{-1} + T\left(\int_{c_x} T^{-1}FT\right)T^{-1}.$$

Proof: Plugging in the vector $\partial/\partial r$, we get

$$A\left(\frac{\partial}{\partial r}\right) = -\frac{\partial T}{\partial r}T^{-1},$$

which follows from the definition of T The vector $\partial/\partial\theta$ gives

$$A_2 = -\frac{\partial T}{\partial \theta}T^{-1} + T\left(\int_0^r T^{-1}F_0T\,dr\right)T^{-1},$$

where

$$A = A_1\,dr + A_2\,d\theta,$$

$$F = F_0 dr \wedge d\theta.$$

To prove this, we first observe that

$$-\frac{\partial A_1}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{\partial T}{\partial r}T^{-1}\right)$$

$$= \frac{\partial^2 T}{\partial\theta\,\partial r}T^{-1} - \frac{\partial T}{\partial r}T^{-1}\frac{\partial T}{\partial \theta}T^{-1}.$$

Hence

$$\frac{\partial}{\partial r}\left[T^{-1}\left(A_2 + \frac{\partial T}{\partial \theta}T^{-1}\right)T\right]$$

$$= -T^{-1}\frac{\partial T}{\partial r}T^{-1}A_2T + T^{-1}\frac{\partial A_2}{\partial r}T + T^{-1}A_2\frac{\partial T}{\partial r}$$

$$- T^{-1}\frac{\partial T}{\partial r}T^{-1}\frac{\partial T}{\partial \theta} + T^{-1}\frac{\partial^2 T}{\partial r\,\partial \theta}$$

$$= T^{-1}A_1A_2T + T^{-1}\frac{\partial A_2}{\partial r}T$$

$$- T^{-1}A_2A_1T - T^{-1}\frac{\partial A_1}{\partial \theta}T$$

$$= T^{-1}F_0T.$$

Continuity of A and $dT \cdot T^{-1}$ at 0 require that

$$\lim_{r\to 0} A_2 = 0,$$

$$\lim_{r\to 0}\frac{\partial T}{\partial \theta}T^{-1} = 0,$$

so

$$T^{-1}\left(A_2 + \frac{\partial T}{\partial \theta}T^{-1}\right)T = \int_0^r T^{-1}F_0T\,dr$$

which gives the formula we wanted.

Theorem 3.2: The holonomy from 0 to 0 along ∂M is given by the θ-ordered exponential

$$P\exp\left(-\int_M T^{-1}FT\right).$$

Proof: Let

$$V(r,\theta) = P\exp\left(\int_0^\theta \int_0^r T^{-1}F_0T\,dr\,d\theta\right)$$

with the ordering over θ, so

$$V(r_0, \theta_0) = P\exp\left(-\int_M T^{-1}FT\right).$$

By Theorem 3.1,

$$V = P\exp\left[-\int_0^{\theta_0}\left(T^{-1}A_2T + T^{-1}\frac{\partial T}{\partial \theta}\right)d\theta\right].$$

We compute

$$\frac{\partial}{\partial \theta}(TV)(TV)^{-1} = T\frac{\partial V}{\partial \theta}V^{-1}T^{-1} + \frac{\partial T}{\partial \theta}T^{-1}$$

$$= -T\left(T^{-1}A_2T + T^{-1}\frac{\partial T}{\partial \theta}\right)T^{-1}$$

$$+ \frac{\partial T}{\partial \theta}T^{-1}$$

$$= -A_2,$$

so by the uniqueness of solutions to this ordinary differential equation, there must be some $W: M \to G$ independent of θ such that

$$TV = P\exp\left(-\int_0^\theta A_2\,d\theta \cdot W\right).$$

Writing $T = T(r, \theta)$ and letting $\theta = 0$ we find that $W = T(r, 0)$, so

$$T(r,\theta)V(r,\theta) = P\exp\left(-\int_0^\theta A_2\,d\theta\,T(r,0)\right).$$

Setting $r = r_0$ and $\theta = \theta_0$ gives

$$V(r_0, \theta_0) = T(r_0, \theta_0)^{-1}P\exp\left(-\int_0^{\theta_0} A_2\,d\theta\,T(r_0, 0)\right)$$

which is precisely the holonomy from 0 to 0 along ∂M.

Proposition 3.3. For any one-form B and any curve c,

(i) $\left|P\exp\left(\int_c B\right)\right| \leqslant \exp\left(\int_c |B|\right)$

(ii) $\left|1 - P\exp\left(\int_c B\right)\right| \leqslant \int_c |B|\exp\left(\int_c |B|\right).$

Proof: Straightforward.

Corollary 3.4: If H is the holonomy along ∂M then

$$|1 - H| \leqslant \int_M |F| \cdot \exp\int_M |F|.$$

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4. ESTIMATES IN THE RADIAL GAUGE

We now choose a gauge in which there is no holonomy in the radial direction. This means that $A(\partial/\partial r) = 0$ where r is the distance to the origin in $\mathbb{R}^4$. Define G: $\mathbb{R}^4 \to U(N)$ by

$$G(x) = P\exp\left(-\int_{c_x} A\right),$$

where

$$c_x(t) = tx, \quad 0 \leqslant t \leqslant 1.$$

Proposition 4.1: (i) Replacing A by $G^{-1}AG + G^{-1}dG$ (and F by $G^{-1}FG$) gives us a radial gauge.

(ii) In this gauge,

$$A(x) = \int_{c_x} F.$$

(iii) If C is a constant such that

$$|F| \leqslant C/r^2 \ln r$$

for $r \geqslant 2$ and

$$|A| \leqslant (C/e) \ln \ln 3$$

for $r = e = 2.718\cdots$, then for $r \geqslant 3$ we have

$$|A| \leqslant 2C(\ln \ln r)/r.$$

Proof: (i) and (ii) follow from Theorem 3.1. For (iii), write $F = F_0 dr \wedge d\theta$, so

$$A = \left(\int_0^r F^0\, dr\right) d\theta.$$

Since $|d\theta| = 1/r$ and $|F| = |F_0|/r$, we have

$$|A| \leqslant \left|\int_0^e F_0\, dr\right| |d\theta| + \left|\int_e^r F_0\, dr\right| |d\theta|$$
$$\leqslant \frac{C}{r} \ln \ln 3 + \frac{1}{r}\int_e^r \frac{C}{r^2 \ln r}\, r\, dr$$
$$= (C/r)\ln \ln 3 + (C/r)\ln \ln r$$
$$\leqslant 2C(\ln \ln r)/r.$$

Proposition 4.2: For $r \geqslant 3$, there exists a constant K_1 such that

$$\left|\int_{S_r^3} \mathrm{tr} AF\right| \leqslant K_1 \cdot \frac{\ln \ln r}{\ln r}.$$

Proof: The volume of S_r^3 is $2\pi r^3$, so

$$\left|\int_{S_r^3} \mathrm{tr} AF\right| \leqslant \int_{S_r^3} |\mathrm{tr}\, AF|$$
$$\leqslant N\int_{S_r^3} |A|\,|F|$$
$$\leqslant N\, 2\pi r^3\, 2C\,[(\ln \ln r)/r](C/r^2 \ln r)$$
$$= 4\pi N C^2 \frac{\ln \ln r}{\ln r}.$$

Proposition 4.3: Suppose $r \geqslant 3$. If

$$|A - B| \leqslant \frac{2C}{r \ln r},$$

then there exists a constant K_2 such that

$$\left|\int_{S_r^3} \mathrm{tr} A^3 - \int_{S_r^3} \mathrm{tr} B^3\right| \leqslant K_2 \cdot \frac{(\ln \ln r)^2}{\ln r}.$$

Proof: We have

$$|B| \leqslant |B - A| + |A|$$
$$\leqslant 2C/r \ln r + 2C(\ln \ln r)/r$$
$$\leqslant 4C(\ln \ln r)/r$$

and

$$A^3 - B^3 = A^2(A - B) + A(A - B)B + (A - B)B^2,$$

so

$$|A^3 - B^3| \leqslant \frac{2C}{r \ln r}\, 3\left(4C \frac{\ln \ln r}{r}\right)^2$$
$$= \frac{96 C^3 (\ln \ln r)^2}{r^3 \ln r}.$$

Thus

$$\left|\int_{S_r^3} \mathrm{tr} A^3 - \int_{S_r^3} \mathrm{tr} B^3\right| \leqslant N\int_{S_r^3} |A^3 - B^3|$$
$$\leqslant N\, 2\pi r^3\, 96 C^3 (\ln \ln r)^2 / r^3 \ln r.$$

5. THE HOLONOMY AT INFINITY

In this section, we restrict attention to S_r^3 for sufficiently large r. $C_1, C_2, \ldots, C_{13}$ will be constants independent of r.

Fix some "north pole" $p \in S_r^3$, and some great circle Γ from p to $-p$. If $x \in S_r^3$ is not on the "equator" S_r^2, let c_x be the shorter great arc from $\pm p$ to x. (There is a unique great circle in S_r^3 pasing through $\pm p$ and x if $x \neq \pm p$.) For x in the "northern hemisphere," let $T(x)$ be the holonomy along c_x. For x in the "southern hemisphere", let $T(x)$ be the holonomy along Γ concatenated with c_x. T is discontinuous on the equator S_r^2.

Lemma 5.1: There exists a constant C_1 such that if T_1, T_2 are the two limiting values of T at a point x in S_r^2, then

$$|T_1 - T_2| \leqslant C_1/\ln r.$$

Proof: Choose a surface S in S_r^3 having area $\leqslant 2\pi r^2$ and boundary the union of Γ with the great semicircle from p to $-p$ through x. Then

$$\int_S |F| \leqslant 2\pi r^2\, C/r^2 \ln r = 2\pi C/\ln r.$$

The holonomy around ∂S is $T_1 T_2^{-1}$, so by Corollary 3.4,

$$|T_1 - T_2| = |1 - T_1 T_2^{-1}| \leqslant \int_S |F| \exp \int_S |F|.$$

We now choose C_1 so that for large r,

$$\frac{2\pi C}{\ln r} \exp\left(\frac{2\pi C}{\ln r}\right) \leqslant \frac{C_1}{\ln r}.$$

Let t: $S_r^3 \to \mathbb{R}$ be the distance to p along S_r^3. Let $\theta = (\theta^1, \theta^2)$ be local coordinates on S_r^2, extended to S_r^3 by requiring θ to be constant along each c_x.

Lemma 5.2: For any $x \in S_r^3$,

$$\left|\int_{c_x} T^{-1} F T\right| \leqslant \frac{2C}{r \ln r}.$$

Proof: We suppose that x is on S_r^2, that c_x is taken to be the great arc from p to x, and that $d\theta^1, d\theta^2$ are orthonormal at x. The general case will fillow easily. Note that

$$|d\theta^1| = \csc(t/r), \quad |dt| = 1.$$

Let

$$F = F_{01} dt \wedge d\theta^1 + F_{02} dt \wedge d\theta^2 + F_{12} d\theta^1 \wedge d\theta^2,$$

so

$$\int_{C_x} T^{-1} F T = \sum_{i=1}^{2} \left(\int_{C_x} T^{-1} F_{0i} T \, dt \right) d\theta^i.$$

Thus

$$\left| \int_{C_x} T^{-1} F T \right| \leqslant \sum_{i=1}^{2} \int_{C_x} |F_{0i}| \, dt$$
$$\leqslant 2 \int_0^{\pi r/2} |F| \sin\left(\frac{t}{r}\right) dt$$
$$\leqslant 2 \frac{C}{r^2 \ln r} \cdot r.$$

If x is a point in the northern hemisphere (i.e., $t < \pi r/2$), then we have by Theorem 3.1,

$$A(x) = -dT\,T^{-1} + T\left(\int_{C_x} T^{-1} F T \right) T^{-1}. \qquad (5.3)$$

If T_0 is constant, the substituting TT_0 for T leaves (5.3) unchanged. It follows that (5.3) is also valid if x is in the southern hemisphere. Hence we have

Corollary 5.4: (i). $|A + dT\,T^{-1}| \leqslant 2C/r \ln r$. (ii) $|dT| \leqslant C_2 (\ln \ln r)/r$ for some constant C_2.

Given $\delta \in (0, 1)$, let R be the set of points in S_r^3 with $\frac{1}{2}\pi r - \delta < t < \frac{1}{2}\pi r + \delta$, and let

$$\mathrm{U}(\theta) = \lim_{t \to \frac{1}{2}\pi r -} T(t, \theta).$$

From Corollary 5.4(ii), it follows that there exists a constant C_3 such that the inequality

$$|dU| \leqslant C_3 (\ln \ln r)/r \qquad (5.5)$$

holds on R. By Lemma 5.1, we can choose δ sufficiently small that on R,

$$|U - T| \leqslant 2C_1/\ln r.$$

We suppose that $2C_1/\ln r < \frac{1}{2}$ so that $f: R \to u(N)$ may be defined by the power series

$$f = \ln U^{-1} T = -\sum_{n=1}^{\infty} \frac{1}{n} (1 - U^{-1} T)^n.$$

Then

$$|f| \leqslant \sum_{n=1}^{\infty} \frac{1}{n} |1 - U^{-1} T|^n$$
$$\leqslant \sum_{n=1}^{\infty} \left(\frac{2C_1}{\ln r} \right)^n$$
$$\leqslant \frac{4C_1}{\ln r}. \qquad (5.6)$$

Let $\phi: \mathbb{R} \to [0, 1]$ be a smooth function satisfying

$$\phi(t) = \begin{cases} 1 & |t| \geqslant 1, \\ 0 & |t| \leqslant \frac{1}{2}. \end{cases}$$

Define $\tilde{T}: R \to \mathrm{U}(N)$ by

$$\tilde{T}(t, \theta) = \mathrm{U}(\theta) \exp[\phi((t - \tfrac{1}{2}\pi r)/\delta) f(t, \theta)]$$

and extend $\tilde{T}$ to be a smooth function on S_r^3 by setting it equal to T on $S_r^3 - R$.

Lemma 5.7: Let V and E be matrix valued functions.

(i) If $V = e^E$ then $|dV| \leqslant e^{|E|} |dE|$,

(ii) If $|1 - V| \leqslant \mu < 1$ and $E = \ln V$, then

$$|dE| \leqslant \frac{1}{1 - \mu} |dV|.$$

Proof: This follows from differentiating the power series for exp and ln.

Lemma 5.8: There exist constants C_4, C_6, C_7 such that on R,

(i) $\quad |df| \leqslant C_4 \dfrac{\ln \ln r}{r}$,

(ii) $\quad \left| \dfrac{\partial \tilde{T}}{\partial t} dt \right| \leqslant |d\tilde{T}| \leqslant \dfrac{C_6}{\delta \ln r}$,

(iii) $\quad \left| \displaystyle\sum_{i=1}^{2} \frac{\partial \tilde{T}}{\partial \theta^i} d\theta^i \right| \leqslant C_7 \dfrac{\ln \ln r}{r}$.

Proof: (i) Apply Lemma 5.7(ii) with $\mu = \frac{1}{2}$. (ii) By Lemma 5.7(i),

$$|d\tilde{T}| \leqslant |dU| e^{|f|} + e^{|f|} |df| + e^{|f|} \left| d\left[\phi\left(\frac{t - \frac{1}{2}\pi r}{\delta} \right) \right] \right| |f|.$$

Then, using (5.5), (5.6), and Lemma 5.8(i),

$$|d\tilde{T}| \leqslant e^{4C_1/\ln r} \left(C_3 \frac{\ln \ln r}{r} + C_4 \frac{\ln \ln r}{r} + \frac{C_5}{\delta} \frac{4C_1}{\ln r} \right)$$
$$\leqslant \frac{C_6}{\delta \ln r}.$$

(iii) Similar, except that the term involving δ is absent.

Lemma 5.9: For some C_9,

$$\left| \int_{S_r^3} \mathrm{tr}(T^{-1} dT)^3 - \int_{S_r^3} \mathrm{tr}(\tilde{T}^{-1} d\tilde{T})^3 \right| \leqslant C_9 \frac{(\ln \ln r)^2}{\ln r}.$$

Proof:

$$\left| \int_{S_r^3} \mathrm{tr}(T^{-1} dT)^3 - \int_{S_r^3} \mathrm{tr}(\tilde{T}^{-1} d\tilde{T})^3 \right|$$
$$= \left| \int_R \mathrm{tr}(T^{-1} dT)^3 - \int_R \mathrm{tr}(\tilde{T}^{-1} d\tilde{T})^3 \right|$$
$$\leqslant N \int_R |dT|^3 + \int_R |\mathrm{tr}(\tilde{T}^{-1} d\tilde{T})^3|.$$

By Lemma 5.8(ii) and 5.8(iii), there exists C_8 such that

$$|\mathrm{tr}(\tilde{T}^{-1} d\tilde{T})^3| \leqslant C_8 (1/\delta \ln r)[(\ln \ln r)/r]^2.$$

The volume of R is bounded by $4\pi r^2 \delta$, so from Corollary 5.4(ii) and the above,

$$\left| \int_{S_r^3} \mathrm{tr}(T^{-1} dT)^3 - \int_{S_r^3} \mathrm{tr}(\tilde{T}^{-1} d\tilde{T})^3 \right|$$
$$\leqslant N\, 4\pi r^2 \delta\, [(C_2 \ln \ln r)/r]^3 + 4\pi r^2 \delta\, C_8\, (1/\delta \ln r)[(\ln \ln r)/r]^2$$
$$\leqslant C_9 (\ln \ln r)^2 / \ln r.$$

We now complete the proof of Theorem 1.1. From $\mathrm{tr} F^2 = d\, \mathrm{tr}(AF - \frac{1}{3} A^3)$ and

$$|F| \leqslant C/r^2 \ln r,$$

it follows that

$$\left|\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\mathrm{tr}F^2-\frac{1}{8\pi^2}\int_{S_r^3}\mathrm{tr}(AF-\tfrac{1}{3}A^3)\right|$$
$$\leq\frac{N}{8\pi^2}\int_r^\infty\left(\frac{C}{\rho^2\ln\rho}\right)^2 2\pi\rho^3\,d\rho=\frac{C_{10}}{\ln r}.$$

From Proposition 4.2,

$$\left|\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\mathrm{tr}F^2+\frac{1}{24\pi^2}\int_{S_r^3}\mathrm{tr}A^3\right|\leq C_{11}\frac{\ln\ln r}{\ln r}.$$

Corollary 5.4(i) shows that the hypothesis of Proposition 4.3 is satisfied if $B=-dT\cdot T^{-1}$, so

$$\left|\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\mathrm{tr}F^2-\frac{1}{24\pi^2}\int_{S_r^3}\mathrm{tr}(T^{-1}dT)^3\right|\leq C_{12}\frac{(\ln\ln r)^2}{\ln r}.$$

By Lemma 5.9,

$$\left|\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\mathrm{tr}F^2-n\right|\leq C_{13}\frac{(\ln\ln r)^2}{\ln r},$$

where n is the integer

$$\frac{1}{24\pi^2}\int_{S_r^3}\mathrm{tr}(\tilde{T}^{-1}d\tilde{T})^3.$$

The proof of Theorem 1.1 is now finished by letting r tend to infinity

6. AN EXAMPLE

Let $T: S^3\to SU(2)$ be the standard identification. Let ω be the pull-back to $\mathbb{R}^4$ of $T^{-1}\,dT$ by the radial projection. The structural equation is

$$d\omega+\omega^2=0.$$

Let $f: [0,\infty)\to\mathbb{R}$ be a smooth function satisfying

$$f(r)=\begin{matrix}0 & 0\leq r\leq\frac{1}{2},\\ a & r\geq 1,\end{matrix}$$

for some real number a. Using r for the radial coordinate, $f(r)\omega$ is a well-defined smooth form on $\mathbb{R}^4$, and we define it to be a connection on the trivial SU(2) bundle.

We compute

$$F=f'dr\wedge\omega+(f^2-f)\omega^2,$$

so $F\in O(1/r^2)$. Also,

$$F^2=2(f^2-f)f'\,dr\wedge\omega^3$$

has compact support and

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\mathrm{tr}F^2=\frac{1}{8\pi^2}\int_0^\infty 2f'(r)(f^2(r)-f(r))\,dr\int_{S^3}\mathrm{tr}\omega^3$$
$$=\frac{1}{8\pi^2}\left[\tfrac{2}{3}f^3(r)-f^2(r)\right]_0^\infty\cdot 12\text{ volume }S^3$$
$$=2a^3-3a^2.$$

By varying a, we may obtain any real number.

7. A GENERALIZATION

Theorem 1.1 generalizes to the following situation. Let G be a compact Lie group and let $\rho: G\to U(N)$ be a representation. Let F be the curvature of some connection on some principal G bundle over $\mathbb{R}^{2n}$. Let

$$c_n=[(-1)^{n+1}/n]\,\mathrm{tr}(\rho(F)/2\pi i)^n.$$

If the connection extends to a connection on some bundle over S^{2n}, then the Chern number[10] of the associated vector bundle is $\int_{\mathbb{R}^{2n}}c_n$. This is an integer and is always divisible by $(n-1)!$ (see Ref. 7 p. 77 or Ref. 11 p. 156).

Theorem 7.1: Suppose $n>1$. If there exists a constant C such that

$$|F|\leq\frac{C}{r^2\ln r},\quad r\geq 2,$$

then

$$\frac{1}{(n-1)!}\int_{\mathbb{R}^{2n}}c_n$$

is an integer. Any integer value is possible if $G=SU(N)$ and $N\geq n$.

Proof: The proof requires only minor modifications of the proof of Theorem 1.1, which we now discuss. According to the Chern–Weil theory (Ref. 7, p. 114),

$$\mathrm{tr}F^n=nd\int_0^1\mathrm{tr}\{A\,[tF+(t^2-t)A^2]^{n-1}\}dt.$$

It follows as before that in the radial gauge,

$$\int_{\mathbb{R}^4}\mathrm{tr}F^n=n\int_0^1(t^2-t)^{n-1}\,dt\,\lim_{r\to\infty}\int_{S_r^{2n-1}}\mathrm{tr}A^{2n-1}.$$

An elementary integration by parts gives

$$\int_0^1 t^{n-1}(1-t)^{n-1}\,dt=\frac{[(n-1)!]^2}{(2n-1)!}.$$

Approximating A by $-d\tilde{T}\,\tilde{T}^{-1}$ for some smooth $\tilde{T}$: $S_r^3\to U(N)$, we find that

$$\frac{1}{(n-1)!}\int_{\mathbb{R}^{2n}}c_n$$

is approximated by

$$\frac{-1}{(2\pi i)^n}\,\frac{(n-1)!}{(2n-1)!}\int_{S_r^{2n-1}}\mathrm{tr}(\tilde{T}^{-1}\,d\tilde{T})^{2n-1}.$$

This is an integer because it is the integral that gives Bott periodicity,[8]

$$\pi_{2n-1}U(N)\simeq\mathbb{Z},$$

for $n\leq N$. The integral is zero if $n>N$. ∎

The proof of Theorem 7.1 breaks down if $n=1$. In order to estimate the holonomy around a closed curve near infinity, we expressed the curve as the boundary of a surface where the curvature was small and used Corollary 3.4. But the sphere of radius r in $\mathbb{R}^{2n}$ is simply connected only if $n>1$.

The following theorem shows that Theorem 7.1 actually fails if $n=1$.

Theorem 7.2: Let F be a two-form on $\mathbb{R}^2$ with values in some Lie algebra. Then F is the curvature of some connection on a principal bundle over $\mathbb{R}^2$.

Proof: Write

$$F=f(x,y)\,dx\wedge dy.$$

Then F is the curvature of the connection

$$A(x,y)=\left(\int_0^x f(x',y)\,dx'\right)dy.$$

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